§3. Basics on automorphic forms

Motivation

analysis		algebra
$\mathbb{R}/\mathbb{Z}$:	$L^2(\mathbb{R}/\mathbb{Z})$, $C(\mathbb{R}/\mathbb{Z})$, $C^\infty(\mathbb{R}/\mathbb{Z})$, $H^d(\mathbb{R}/\mathbb{Z})$	{trigonometric polynomials} $=$ {finite linear combinations of $x \mapsto e^{2\pi i n x}$}
$\mathbb{R}$:	$(\dots)$, $\mathcal{S}(\mathbb{R})$	· $[x \mapsto e^{ix\xi}]$, finite lin. comb. $\mid \frac{d}{dx} e^{ix\xi} = i\xi e^{ix\xi}$ · $[x \mapsto P(x) e^{-\pi x^2}]$, P: polynomial (or Hermite functions)
$\Gamma \backslash G = SL_n\mathbb{Z} \backslash SL_n\mathbb{R}$ or $GL_n\mathbb{Z} \backslash GL_n\mathbb{R}$	$K = SO(n)$ or $O(n)$	automorphic forms

Definition (to be explained) An automorphic form $\varphi : G \to \mathbb{C}$ is a continuous function such that

(A1) $\varphi(\gamma x) = \varphi(x)$ $\forall \gamma \in \Gamma$, $x \in G$ $\quad (\varphi : \Gamma \backslash G \to \mathbb{C})$

(A2) φ is right K-finite : span$\{\varphi(\cdot k) : k \in K\}$ is finite-dim'l

(A3) φ is $\mathfrak{Z}(\mathfrak{g})$-finite, $\mathfrak{Z}(\mathfrak{g})$ = center of universal enveloping algebra

(A4) φ is of moderate growth.

Norms on $G \ni x$ $\quad \|x\|^2 := \sum_{i,j} \left(|x_{ij}|^2 + |(x^{-1})_{ij}|^2 \right)$

NB $\left\| \begin{pmatrix} \varepsilon & & \\ & \ddots & \\ & & \varepsilon \end{pmatrix} \right\| \sim 1/\varepsilon$ as $\varepsilon \to 0$

Basic properties
- $\|xy\| \le \|x\| \cdot \|y\|$, $\quad \|x^{-1}\| = \|x\|$
- $\{x \in G : \|x\| \le C\}$ is compact
- $\forall$ compact $\Omega \subseteq G$, $\|u x u'\| \asymp \|x\|$ $\forall u, u' \in \Omega$, $x \in G$

Defn $\varphi : G \to \mathbb{C}$ is of moderate growth if $\exists m : \varphi(x) \ll \|x\|^m$ $\forall x \in G$

—— // —— rapid decay if $\forall m$: —— // ——

Finiteness conditions

<u>Lemma</u> Let G: locally compact group, $\varphi : G \to \mathbb{C}$ continuous.
The following are equivalent:

(i) $\dim \operatorname{span} \{ \varphi(\cdot\, g) : g \in G \} < \infty$

(ii) $\dim \operatorname{span} \{ \varphi(g\, \cdot) : g \in G \} < \infty$

(iii) $\dim \operatorname{span} \{ \varphi(g_1 \cdot g_2) : g_1, g_2 \in G \} < \infty$

(iv) φ arises as a matrix coefficient of a finite-dimensional representation (π, V) of G: $\exists\, \alpha \in \operatorname{End}(V)^*$ s.t. $\varphi(g) = \alpha(\pi(g))$.

<u>Proof idea</u> Assume (i). Consider $W := \operatorname{span}\{\varphi(\cdot\, g) : g \in G\}$, $\ell : W \to \mathbb{C}$, $f \mapsto f(1)$.
$\ell \in W^*$. Show ℓ: G-finite (References in notes.) $\square$

<u>Defn</u> φ is <u>finite</u> if it satisfies these (equivalent) conditions.

<u>Examples</u> $G = \mathbb{R}/\mathbb{Z} \Rightarrow \{\text{finite functions}\} = \{\text{trigonometric polynomials}\}$

<u>Examples</u> $G = \mathbb{R} \Rightarrow \{\text{finite functions}\} = \{\text{exponential polynomials}\}$

$$:= \left\{ \begin{matrix} \text{finite linear combinations of functions of the form} \\ x \mapsto e^{\alpha x} x^{\beta}, \quad \alpha \in \mathbb{C},\ \beta \in \mathbb{Z}_{\geq 0} \end{matrix} \right\}.$$

<u>Example</u> $G = \mathbb{R}_+^\times \xrightarrow{\log} \mathbb{R}$, $\{\text{finite functions}\} = \{\text{exponential polynomials}\}$

$$:= \langle y \mapsto y^{\alpha} (\log y)^{\beta} : \text{—〃—} \rangle$$

<u>Example</u> G = compact group. $\underbrace{\{\text{finite functions}\} = \text{span of matrix coeffs. of irreducible reps.}}$

Peter-Weyl Theorem: $\{\text{finite functions}\}$ is dense in $L^2(G)$, $C(G)$, $C^\infty(G)$.

<u>Lie-theoretic basics</u> G: Lie group, assume $G \leq GL_n(\mathbb{R})$
$\mathfrak{g} = \operatorname{Lie}(G) \subseteq M_n(\mathbb{R})$

$G \circlearrowright$ itself by left + right translation, hence also on functions $\varphi : G \to \mathbb{C}$:

$$\ell_g \varphi(x) := \varphi(g^{-1} x), \qquad r_g \varphi(x) = \varphi(xg).$$

$\Rightarrow \ell_{g_1} \ell_{g_2} = \ell_{g_1 g_2}$, $r_{g_1} r_{g_2} = r_{g_1 g_2}$ (Exercise!)

$$\mathfrak{g} \ni X \rightsquigarrow e^X, \quad e^{tX} = \sum_{n \geq 0} t^n \frac{X^n}{n!}$$

$G \circlearrowright \mathfrak{g}$ via $\operatorname{Ad} : G \to GL(\mathfrak{g})$ $\qquad g e^{tX} g^{-1} = e^{t \operatorname{Ad}(g) X}$

$$\operatorname{Ad}(g) X = g X g^{-1}$$

Universal enveloping algebra ($\mathfrak{g}$: Lie algebra)

Defn The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a unital associative algebra equipped with a morphism of Lie algebras $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that $\forall$ unital associative algebras A equipped with morphisms of Lie algebras $\mathfrak{g} \to A$ $\exists!$ algebra morphism $\mathcal{U}(\mathfrak{g}) \to A$ s.t.

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & \mathcal{U}(\mathfrak{g}) \arrow[d, dotted] \\ & A \end{tikzcd}$$

commutes.

Construction 1 $\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/I$,

$$T(\mathfrak{g}) := \text{tensor algebra} := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{g}^{\otimes k}$$

$$I = \langle X \otimes Y - Y \otimes X - [X, Y] : X, Y \in \mathfrak{g} \rangle$$

Construction 2 Suppose $\mathfrak{g} = \mathrm{Lie}(G)$. Then G acts on $C^\infty(G)$ by right translation: $r_g \varphi(x) = \varphi(xg)$. This defines

$$G \longrightarrow GL(C^\infty(G)).$$

It differentiates to

$$\mathfrak{g} \longrightarrow \mathrm{End}(C^\infty(G))$$

$$X \longmapsto \Big[\varphi \longmapsto \underbrace{\partial_{t=0}\, r_{\exp(tX)} \varphi}\Big]$$

$$= \big[x \mapsto \partial_{t=0}\, \varphi(x e^{tX})\big].$$

Then $\mathcal{U}(\mathfrak{g}) \cong$ subalgebra of $\mathrm{End}(C^\infty(G))$ generated by $\mathfrak{g}$.

$$\mathfrak{g} \cong \{\text{left-inv. } \underbrace{\text{vector fields}}_{\text{linear diff. ops on } C^\infty(G)} \text{ on } G\}$$

$$\mathcal{U}(\mathfrak{g}) \cong \{\text{left-inv. diff. ops on } C^\infty(G)\}$$

as in Construction 2

Defn $\mathcal{Z}(\mathfrak{g}) :=$ center of $\mathcal{U}(\mathfrak{g})$. $(A_3) \overset{\text{DEF}}{\Longleftrightarrow} \dim(\mathcal{Z}(\mathfrak{g}) \cdot \varphi) < \infty$

Fact For $G = GL_n(\mathbb{R})$, $\mathcal{Z}(\mathfrak{g})$ is f.g. alg, $\cong \mathbb{R}[\underbrace{x_1, \dots, x_n}_{\text{independent variables}}]$

Example $G = SL_2(\mathbb{R})$ $\qquad \mathfrak{Z}(\mathfrak{g}) = \mathbb{R}[\Omega]$,

where $\Omega = ef + fe + \frac{1}{2}h^2$,

mult. in $\mathcal{U}(\mathfrak{g})$

$$e = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad f = \begin{pmatrix} & \\ 1 & \end{pmatrix}, \quad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Relation to classical modular forms $\quad G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, $K = SO(2)$

$$G/K \cong \mathbb{H} := \{x+iy : y>0\}$$
$$gK \mapsto g\cdot i, \qquad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot z = \frac{az+b}{cz+d}$$

Defn A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight m if:

(M_1) $\quad f(\gamma z) = (cz+d)^m f(z) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ z \in \mathbb{H}$.

(M_2) $\quad f$ is holomorphic $\qquad$ (Serre, Course in Arithmetic)

(M_3) $\quad f$ is "regular at the cusp ∞"

Explanation of (M_3): $\quad (M_1) \Rightarrow f(z+1) = f(z)$

$$(M_2) \Rightarrow f(z) = \sum_{n\in\mathbb{Z}} a_n \underbrace{e^{2\pi i n z}}_{e^{2\pi i n x}\underbrace{e^{-2\pi n y}}_{\text{blows up if } n<0}}$$

$$(M_3) \overset{\text{DEFN}}{\Longleftrightarrow} a_n = 0 \ \text{unless} \ n \geq 0.$$

Relation Define $\varphi : G \to \mathbb{C}$ by $\varphi(g) := (ci+d)^{-m} f(g\cdot i)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then $(M_1) \Rightarrow (A_1)$ (exercise)

$(M_1) \Rightarrow (A_2)$, in fact $\varphi\left(g\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{2\pi i m\theta}\varphi(g)$

$\Rightarrow \dim \operatorname{span}\{\varphi(\cdot k) : k \in K\} = 1$

$(M_2) \Rightarrow (A_3)$, in fact $\Omega\varphi = c\varphi$, $\quad c \overset{?}{=} (?)\,\frac{m(m+1)}{2}$. (exercise)

$(M_3) \Rightarrow (A_4)$: if $g = \underbrace{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}}_{\in \mathfrak{S} \,:\, \text{Siegel}} k$, then $\|g\| \asymp y^{1/2}$

So (A_4) says: on $\{x+iy : |x| \leq \frac{1}{2}, \ y \geq \frac{\sqrt{3}}{2}\}$,

$$f(x+iy) \ll y^m \ \text{for some } m.$$
$$\Longleftrightarrow a_n = 0 \ \forall\, n < 0.$$

Some basic properties

Thm φ : autom. form $\implies$ φ : analytic ($\implies$ smooth)

Pf idea K-finiteness + $\mathfrak{Z}(\mathfrak{g})$-finiteness conditions imply that φ satisfies some elliptic PDE.

Thm (Harish-Chandra) φ: autom form $\implies$ $\exists\, f \in C_c^\infty(G)$, supported arbitrarily close to the identity element $1 \in G$, such that

$$\varphi * f = \varphi, \qquad \varphi * f(x) := \int_{g \in G} \varphi(x g^{-1}) f(g)\, dg.$$

and $f(k^{-1} g k) = f(g)$ $\forall\, k \in K,\ g \in G$.

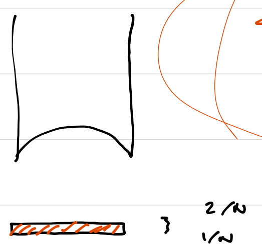
Remark Let f: modular form of weight m.

$$f = \sum_{n \geq 0}^{\infty} a_n \underbrace{e^{2\pi i n z}}_{e^{2\pi i n x}\, \underbrace{e^{-2\pi n y}}_{\text{decays as } y \to \infty \text{ unless } n = 0}} \sim a_0 \quad \text{as } y \to \infty.$$

$a_n \ll n^{O(1)}$

Sketch (Hecke bound) Consider

$$\int_{x=0}^{1} \int_{y=1/N}^{2/N} \underbrace{|f(x+iy)|^2}_{\ll N^{O(1)} \text{ by moderate growth assumption}} dx\, dy = \sum_n |a_n|^2 \underbrace{\int_{y=1/N}^{2/N} e^{-4\pi n y}\, dy}_{\gg 1 \text{ if } n \asymp N}$$



Q What does the H-C theorem above say about φ?

φ: any function on G that is right K-finite and $\mathfrak{Z}(\mathfrak{g})$-finite, then closure(Span $\{\varphi(\cdot g) : g \in G\}$ in $C^\infty(G)$) has the property that each irreducible representation of K occurs in it with finite multiplicity.